

Vertices of Simple Modules of Symmetric Groups Labelled by Hook Partitions

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Abstract

In this article we study the vertices of simple modules for the symmetric groups in prime characteristic p . In particular, we complete the classification of the vertices of simple $F\mathfrak{S}_n$ -modules labelled by hook partitions.

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1 Introduction

Introduced by J. A. Green in 1959 [5], *vertices* of indecomposable modules over modular group algebras have proved to be important invariants linking the global and local representation theory of finite groups over fields of positive characteristic. Given a finite group G and a field F of characteristic $p > 0$, by Green's result, the vertices of every indecomposable FG -module form a G -conjugacy class of p -subgroups of G . Moreover, vertices of simple FG -modules are known to satisfy a number of very restrictive properties, most notably in consequence of Knörr's Theorem [8]. The latter, in particular, implies that vertices of simple FG -modules in blocks with abelian defect groups have precisely these defect groups as their vertices. Despite this result, the precise structure of vertices of simple FG -modules is still poorly understood, even for very concrete groups and modules.

The aim of this paper is to complete the description of the vertices of a distinguished class of simple modules of finite symmetric groups. Throughout, let $n \in \mathbb{N}$, and let \mathfrak{S}_n be the symmetric group of degree n . Then, as is well known, the isomorphism classes of simple $F\mathfrak{S}_n$ -modules are labelled by the p -regular partitions of n . We denote the simple $F\mathfrak{S}_n$ -module corresponding to a p -regular partition λ by D^λ . If $\lambda = (n - r, 1^r)$, for some $r \in \{0, \dots, p - 1\}$, then λ is called a p -regular *hook partition* of n . Whilst, in general, even the dimensions of the simple $F\mathfrak{S}_n$ -modules are unknown, one has a neat description of an F -basis of $D^{(n-r, 1^r)}$; we shall comment on this in 2.3 below.

The problem of determining the vertices of the simple $F\mathfrak{S}_n$ -module $D^{(n-r, 1^r)}$ has been studied before by Wildon in [12], by Müller and Zimmermann in [9], and by the first author in [4]. In consequence of these results, the vertices of $D^{(n-r, 1^r)}$ have been known, except in the case where $p > 2$, $r = p - 1$ and $n \equiv p \pmod{p^2}$. In Section 4 of the current paper we shall now prove the following theorem, which together with [4, Corollary 5.5] proves [9, Conjecture 1.6(a)].

1.1 Theorem. *Let $p > 2$, let F be a field of characteristic p , and let $n \in \mathbb{N}$ be such that $n \equiv p \pmod{p^2}$. Then the vertices of the simple $F\mathfrak{S}_n$ -module $D^{(n-p+1, 1^{p-1})}$ are precisely the Sylow p -subgroups of \mathfrak{S}_n .*

Our key ingredients for proving Theorem 1.1 will be the Brauer construction in the sense of Broué [3] and Wildon's result in [12]. Both of these will enable us to obtain lower bounds

on the vertices of $D^{(n-p+1, 1^{p-1})}$, which together will then provide sufficient information to deduce Theorem 1.1.

To summarize, the abovementioned results in [4, 9, 12] and Theorem 1.1 lead to the following exhaustive description of the vertices of the modules $D^{(n-r, 1^r)}$:

1.2 Theorem. *Let F be a field of characteristic $p > 0$, and let $n \in \mathbb{N}$. Let further $r \in \{0, 1, \dots, p-1\}$, and let Q be a vertex of the simple $F\mathfrak{S}_n$ -module $D^{(n-r, 1^r)}$.*

- (a) *If $p \nmid n$ then Q is \mathfrak{S}_n -conjugate to a Sylow p -subgroup of $\mathfrak{S}_{n-r-1} \times \mathfrak{S}_r$.*
- (b) *If $p = 2$, $p \mid n$ and $(n, r) \neq (4, 1)$ then Q is a Sylow 2-subgroup of \mathfrak{S}_n .*
- (c) *If $p = 2$, $n = 4$ and $r = 1$ then Q is the unique Sylow 2-subgroup of \mathfrak{A}_4 .*
- (d) *If $p > 2$ and $p \mid n$ then Q is a Sylow p -subgroup of \mathfrak{S}_n .*

In the case where $p \nmid n$, the simple module $D^{(n-r, 1^r)}$ is isomorphic to the Specht $F\mathfrak{S}_n$ -module $S^{(n-r, 1^r)}$, by work of Peel [11]. Thus assertion (a) follows immediately from [12, Theorem 2]. Assertions (b) and (c) have been established by Müller and Zimmermann [9, Theorem 1.4]. Moreover, if $p > 2$, $p \mid n$ and $r < p-1$ then assertion (d) can also be found in [9, Theorem 1.2]. The case where $p > 2$, $p \mid n$, $r = p-1$ was treated in [4, Corollary 5.5], except when $n \equiv p \pmod{p^2}$, which is covered by Theorem 1.1 above.

We should also like to comment on the sources of the simple $F\mathfrak{S}_n$ -modules $D^{(n-r, 1^r)}$. For $r = 0$, we get the trivial $F\mathfrak{S}_n$ -module $D^{(n)}$, which has of course trivial source. If $p \mid n$, then the module $D^{(n-1, 1)}$ restricts indecomposably to its vertices, by [9, Theorems 1.3, 1.5], except when $p = 2$ and $n = 4$. For $p = 2$, the simple $F\mathfrak{S}_4$ -module $D^{(3, 1)}$ has trivial source, by [9, Theorem 1.5]. If $p \nmid n$ then $D^{(n-r, 1^r)} \cong S^{(n-r, 1^r)}$ has always trivial sources; see, for instance [9, Theorem 1.3]. However, in the case where $p > 2$, $p \mid n$ and $r > 1$, we do not know the sources of $D^{(n-r, 1^r)}$. In these latter cases, the restrictions of $D^{(n-r, 1^r)}$ to its vertices should, conjecturally, be indecomposable, hence should be sources of $D^{(n-r, 1^r)}$; see [9, Conjecture 1.6(b)]. This conjecture has been verified computationally in several cases, see [4, 9], but remains still open in general.

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2 Prerequisites

Throughout this section, let F be a field of characteristic $p > 0$. We begin by introducing some basic notation that we shall use repeatedly throughout subsequent sections. Whenever G is a finite group, FG -modules are always understood to be finite-dimensional left modules. Whenever H and K are subgroups of G such that H is G -conjugate to a subgroup of K , we write $H \leqslant_G K$. If H and K are G -conjugate then we write $H =_G K$. For $g \in G$, we set ${}^gH := gHg^{-1}$.

We assume the reader to be familiar with the basic concepts of the representation theory of the symmetric groups. For background information we refer to [6, 7]. As usual, for $n \in \mathbb{N}$, we shall denote the Specht $F\mathfrak{S}_n$ -module labelled by a partition λ of n by S^λ . If λ is a p -regular partition of n then we shall denote the simple $F\mathfrak{S}_n$ -module $S^\lambda/\text{Rad}(S^\lambda)$ by D^λ .

2.1. Brauer constructions and vertices. (a) Let G be a finite group, let M be an FG -module, and let P be a p -subgroup of G . The *Brauer construction* of M with respect to P is defined as

$$M(P) := M^P / \sum_{Q < P} \text{Tr}_Q^P(M^Q), \quad (1)$$

where M^P denotes the set of P -fixed points of M , and $\text{Tr}_Q^P : M^Q \rightarrow M^P$, $m \mapsto \sum_{xQ \in P/Q} xm$ denotes the relative trace map. The latter is independent of the choice of representatives of the left cosets P/Q . The FG -module structure of M induces an $FN_G(P)$ -module structure on the F -vector space $M(P)$, and P acts trivially on $M(P)$. Set $\text{Tr}^P(M) := \sum_{Q < P} \text{Tr}_Q^P(M^Q)$.

Moreover, if $R < Q < P$ then $\text{Tr}_R^P = \text{Tr}_Q^P \circ \text{Tr}_R^Q$. Thus $M(P) = M^P / \sum_{Q <_{\max} P} \text{Tr}_Q^P(M^Q)$, where $Q <_{\max} P$ denotes a maximal subgroup of P . If $Q <_{\max} P$ then every element $g \in P \setminus Q$ has the property that $\{1, g, g^2, \dots, g^{p-1}\}$ is a set of representatives of the left cosets of Q in P ; in particular, we get $\text{Tr}_Q^P(m) = m + gm + \dots + g^{p-1}m$, for $m \in M^Q$.

(b) Suppose that M is an indecomposable FG -module. Then a *vertex* of M is a subgroup Q of G that is minimal with respect to the property that M is isomorphic to a direct summand of $\text{Ind}_Q^G(\text{Res}_Q^G(M))$. By [5], the vertices of M form a G -conjugacy class of p -subgroups of G . Moreover, if $R \leq G$ is a p -subgroup such that $M(R) \neq \{0\}$ then $R \leq_G Q$, by [3, (1.3)]. The converse is, however, not true in general.

For proofs of the abovementioned properties of Brauer constructions, see [3]. Details on the theory of vertices of indecomposable FG -modules can be found in [1, Section 9] or [10, Section 4.3]. The following will be very useful for proving Theorem 1.1 in Section 4 below. The proof is straightforward, and is thus left to the reader.

2.2 Proposition. *Let G be a finite group, let M be an FG -module with F -basis B , and let $P \leq G$ be a p -group. Suppose that there is some $b_0 \in B$ satisfying the following properties:*

- (i) $b_0 \in M^P$;
- (ii) *whenever $Q <_{\max} P$, $u \in M^Q$ and $\text{Tr}_Q^P(u) = \sum_{b \in B} a_b(u)b$, for $a_b(u) \in F$, one has $a_{b_0}(u) = 0$.*

Then $b_0 + \text{Tr}^P(M) \in M(P) \setminus \{0\}$.

Next we shall recall some well-known properties of the simple $F\mathfrak{S}_n$ -modules labelled by hook partitions $(n - r, 1^r)$, for $r \in \{0, \dots, p - 1\}$, that we shall need repeatedly in the proof of Theorem 1.1. In particular, we shall fix a convenient F -basis of $D^{(n-r, 1^r)}$. In light of Theorem 1.1 we shall only be interested in the case where $p \mid n$ and $p > 2$.

2.3. Exterior powers of the natural $F\mathfrak{S}_n$ -module. (a) Let $p > 2$, let $n \in \mathbb{N}$ be such that $p \mid n$, and let $M := M^{(n-1, 1)}$ be the natural permutation $F\mathfrak{S}_n$ -module, with natural permutation basis $\Omega = \{\omega_1, \dots, \omega_n\}$. Since $p \mid n$, the module M is uniserial with composition series $\{0\} \subset M_2 \subset M_1 \subset M$, where $M_1 = \{\sum_{i=1}^n a_i \omega_i : a_1, \dots, a_n \in F, \sum_{i=1}^n a_i = 0\}$ and $M_2 = \{a \sum_{i=1}^n \omega_i : a \in F\}$; see, for instance, [6, Example 5.1].

Furthermore, $M_1 = S^{(n-1,1)}$, and $M_1/M_2 =: \text{Hd}(S^{(n-1,1)}) \cong D^{(n-1,1)}$; in particular, $\dim_F(D^{(n-1,1)}) = n-2$. One sometimes calls $D^{(n-1,1)}$ the *natural (simple) $F\mathfrak{S}_n$ -module*. An F -basis of M_1 is given by the elements $\omega_i - \omega_1$, where $i \in \{2, \dots, n\}$. In the following, we shall identify the module $D^{(n-1,1)}$ with M_1/M_2 .

Consider the natural epimorphism $\pi : M_1 \rightarrow M_1/M_2$, and set $e_i := \overline{\omega_i - \omega_1}$, for $i \in \{1, \dots, n\}$. Then $e_n = -e_2 - e_3 - \dots - e_{n-1}$, and the elements e_2, \dots, e_{n-1} form an F -basis of $D^{(n-1,1)}$.

(b) Let $r \in \{0, \dots, n-1\}$. By [9, Proposition 2.2], there is an $F\mathfrak{S}_n$ -isomorphism $S^{(n-r,1^r)} \cong \bigwedge^r S^{(n-1,1)}$. Moreover, if $r \leq n-2$ then, in consequence of [11], $\text{Hd}(\bigwedge^r S^{(n-1,1)}) \cong \bigwedge^r \text{Hd}(S^{(n-1,1)}) \cong \bigwedge^r D^{(n-1,1)} =: D_r$ is simple. Thus D_r has F -basis

$$\mathcal{B}_r := \{e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_r} : 2 \leq i_1 < i_2 < \dots < i_r \leq n-1\}. \quad (2)$$

If $r \leq p-1$ then $\bigwedge^r D^{(n-1,1)} \cong D^{(n-r,1^r)}$.

3 Symmetric Groups and p -Subgroups

Throughout this section, let $n \in \mathbb{N}$, and let p be a prime number. Permutations in the symmetric group \mathfrak{S}_n will be composed from right to left, so that, for instance, we have $(1,2)(2,3) = (1,2,3) \in \mathfrak{S}_3$.

3.1 Definition. Given an element $\sigma \in \mathfrak{S}_n$, we call $\text{supp}(\sigma) := \{i \in \{1, \dots, n\} : \sigma(i) \neq i\}$ the *support* of σ . If $H \leq \mathfrak{S}_n$ then we call $\text{supp}(H) := \bigcup_{\sigma \in H} \text{supp}(\sigma)$ the *support* of H .

3.2. Sylow subgroups of symmetric groups. (a) Let P_p be the cyclic group $\langle (1, 2, \dots, p) \rangle \leq \mathfrak{S}_p$ of order p . Let further $P_1 := \{1\}$ and, for $d \geq 1$, we set

$$P_{p^{d+1}} := P_{p^d} \wr P_p := \{(\sigma_1, \dots, \sigma_p; \pi) : \sigma_1, \dots, \sigma_p \in P_{p^d}, \pi \in P_p\}.$$

Recall that, for $d \geq 2$, the multiplication in P_{p^d} is given by $(\sigma_1, \dots, \sigma_p; \pi)(\sigma'_1, \dots, \sigma'_p; \pi') = (\sigma_1 \sigma'_{\pi^{-1}(1)}, \dots, \sigma_p \sigma'_{\pi^{-1}(p)}; \pi \pi')$, for $(\sigma_1, \dots, \sigma_p; \pi), (\sigma'_1, \dots, \sigma'_p; \pi') \in P_{p^d}$.

We shall always identify P_{p^d} with a subgroup of \mathfrak{S}_{p^d} in the usual way. That is, $(\sigma_1, \dots, \sigma_p; \pi) \in P_{p^d}$ is identified with the element $\overline{(\sigma_1, \dots, \sigma_p; \pi)} \in \mathfrak{S}_{p^d}$ that is defined as follows: if $j \in \{1, \dots, p^d\}$ is such that $j = p^{d-1}(a-1) + b$, for some $a \in \{1, \dots, p\}$ and some $b \in \{1, \dots, p^{d-1}\}$ then $\overline{(\sigma_1, \dots, \sigma_p; \pi)}(j) := p^{d-1}(\pi(a) - 1) + \sigma_{\pi(a)}(b)$. Via this identification, P_{p^d} can be generated by the elements $g_1, \dots, g_d \in \mathfrak{S}_{p^d}$, where

$$g_j := \prod_{k=1}^{p^{j-1}} (k, k + p^{j-1}, k + 2p^{j-1}, \dots, k + (p-1)p^{j-1}) \quad (1 \leq j \leq d). \quad (3)$$

In particular, with this notation we have $P_p \leq P_{p^2} \leq \dots \leq P_{p^{d-1}} \leq P_{p^d}$, and the base group of the wreath product $P_{p^{d-1}} \wr P_p$ has the form $\prod_{i=0}^{p-1} g_d^i \cdot P_{p^{d-1}} \cdot g_d^{-i}$.

(b) Now let $n \in \mathbb{N}$ be arbitrary, and consider the p -adic expansion $n = \sum_{i=0}^r n_i p^i$ of n , where $0 \leq n_i \leq p-1$ for $i \in \{0, \dots, r\}$, and where we may suppose that $n_r \neq 0$. By [7, 4.1.22, 4.1.24], the Sylow p -subgroups of \mathfrak{S}_n are isomorphic to the direct product $\prod_{i=0}^r (P_{p^i})^{n_i}$. For

subsequent computations it will be useful to fix a particular Sylow p -subgroup P_n of \mathfrak{S}_n as follows: for $i \in \{t \in \mathbb{N} \mid n_t \neq 0\}$ and $1 \leq j_i \leq n_i$, let $k(j_i) := \sum_{l=0}^{i-1} n_l p^l + (j_i - 1)p^i$ and

$$P_{p^i, j_i} := (1, 1 + k(j_i)) \cdots (p^i, p^i + k(j_i)) \cdot P_{p^i} \cdot (1, 1 + k(j_i)) \cdots (p^i, p^i + k(j_i)).$$

Now set

$$P_n := P_{p,1} \times \cdots \times P_{p,n_1} \times \cdots \times P_{p^r,1} \times \cdots \times P_{p^r,n_r}.$$

Given this convention, we shall then also write $P_n = \prod_{i=0}^r (P_{p^i})^{n_i}$, for simplicity.

3.3 Example. Suppose that $p = 3$. Then $P_3 = \langle g_1 \rangle$, $P_9 = \langle g_1, g_2 \rangle$ and $P_{27} = \langle g_1, g_2, g_3 \rangle$, where

$$\begin{aligned} g_1 &= (1, 2, 3), \\ g_2 &= (1, 4, 7)(2, 5, 8)(3, 6, 9), \\ g_3 &= (1, 10, 19)(2, 11, 20)(3, 12, 21)(4, 13, 22)(5, 14, 23)(6, 15, 24)(7, 16, 25)(8, 17, 26)(9, 18, 27). \end{aligned}$$

Moreover, $P_{51} = P_3 \times P_3 \times P_9 \times P_9 \times P_{27}$.

3.4. Elementary abelian groups. (a) Suppose again that $n = p^d$, for some $d \in \mathbb{N}$. We shall denote by E_n the following elementary abelian subgroup of P_n that acts regularly on $\{1, \dots, n\}$: let g_1, \dots, g_d be the generators of P_n fixed in (3). For $j \in \{1, \dots, d-1\}$, let $g_{j,j+1} := \prod_{i=0}^{p-1} g_{j+1}^i g_j g_{j+1}^{-i}$, and for $l \in \{1, \dots, d-j-1\}$, we inductively set

$$g_{j,j+1,\dots,j+l+1} := \prod_{i=0}^{p-1} g_{j+l+1}^i \cdot g_{j,j+1,\dots,j+l} \cdot g_{j+l+1}^{-i}.$$

Then $E_n := \langle g_{1,\dots,d}, g_{2,\dots,d}, \dots, g_{d-1,d}, g_d \rangle$, and $|E_n| = n = p^d$.

(b) Let $n \in \mathbb{N}$ be arbitrary with $p \mid n$, and let $t, m_1, \dots, m_t \in \mathbb{N}_0$ be such that $n = \sum_{i=1}^t m_i p^i$. For $i \in \{s \in \mathbb{N} \mid m_s \neq 0\}$ and $1 \leq j_i \leq m_i$, we set $k(j_i) := \sum_{l=0}^{i-1} m_l p^l + (j_i - 1)p^i$ and

$$E_{p^i, j_i} := (1, 1 + k(j_i)) \cdots (p^i, p^i + k(j_i)) \cdot E_{p^i} \cdot (1, 1 + k(j_i)) \cdots (p^i, p^i + k(j_i)).$$

Then $E(m_1, \dots, m_t) \leq \mathfrak{S}_n$ denotes the elementary abelian group

$$E_{p,1} \times \cdots \times E_{p,m_1} \times \cdots \times E_{p^t,1} \times \cdots \times E_{p^t,m_t}.$$

We emphasize that, unlike in 3.2, the integers m_1, \dots, m_t need not be less than p .

3.5 Example. Suppose that $p = 3$ and $n = 27$. Then $E_n = E_{27}$ is generated by the elements

$$\begin{aligned} g_{1,2,3} &= (1, 2, 3)(4, 5, 6)(7, 8, 9)(10, 11, 12)(13, 14, 15)(16, 17, 18)(19, 20, 21)(22, 23, 24)(25, 26, 27), \\ g_{2,3} &= (1, 4, 7)(2, 5, 8)(3, 6, 9)(10, 13, 16)(11, 14, 17)(12, 15, 18)(19, 22, 25)(20, 23, 26)(21, 24, 27), \\ g_3 &= (1, 10, 19)(2, 11, 20)(3, 12, 21)(4, 13, 22)(5, 14, 23)(6, 15, 24)(7, 16, 25)(8, 17, 26)(9, 18, 27). \end{aligned}$$

We recall the following lemma from [4], which will be useful for our subsequent considerations.

3.6 Lemma ([4, Lemma 2.1, Remark 2.2]). *Let $n \in \mathbb{N}$ with p -adic expansion $n = \sum_{i=0}^r n_i p^i$, as in 3.2. Let $P \leq P_n$ be such that $P = \mathfrak{S}_n P_{p^i}$, for some $i \in \{1, \dots, r\}$. Then $P \leq P_{p^l, j_l}$, for some $l \in \{i, \dots, r\}$ and some $1 \leq j_l \leq n_l$. Moreover, P_{p^l, j_l} has precisely p^{l-i} subgroups that are \mathfrak{S}_n -conjugate to P_{p^i} , and these are pairwise P_{p^l, j_l} -conjugate to each other.*

3.7 Remark. Let again $n \in \mathbb{N}$ with p -adic expansion $n = \sum_{i=0}^r n_i p^i$.

(a) Let $P \leq P_n$ be such that $P = \mathfrak{S}_n P_{p^i}$, for some $i \in \{1, \dots, r\}$, so that $P \leq P_{p^l, j_l}$, for some $l \in \{i, \dots, r\}$ and some $1 \leq j_l \leq n_l$, by Lemma 3.6. Note that the subgroups of P_{p^l, j_l} that are \mathfrak{S}_n -conjugate to P_{p^i} are uniquely determined by their supports. In particular, if $i = 1$ then P is generated by one of the p -cycles $(1, \dots, p), \dots, (n - n_0 - p + 1, \dots, n - n_0) \in P_n$.

(b) Suppose that $E \leq P_n$ is such that $E = \mathfrak{S}_n E_{p^i}$, for some $i \in \{1, \dots, r\}$. Since E has precisely one non-trivial orbit, we then also get $E \leq P_{p^l, j_l}$, for some $l \in \{i, \dots, r\}$ and some $1 \leq j_l \leq n_l$. Moreover, arguing by induction on $l - i$ as in the proof of [4, Lemma 2.1], we deduce that E then has to be contained in one of the p^{l-i} subgroups of P_{p^l, j_l} that are \mathfrak{S}_n -conjugate to P_{p^i} .

3.8 Lemma. *Let $n, d \in \mathbb{N}$, and let $P \leq P_{p^d} \leq \mathfrak{S}_n$. Suppose that P contains an \mathfrak{S}_n -conjugate of $P_{p^{d-1}}$. Suppose further that P contains an elementary abelian group E of order p^d acting regularly on $\{1, \dots, p^d\}$. Then $P = P_{p^d}$.*

Proof. If $d = 1$ then $P_{p^d} = P_p = E$. From now on we may suppose that $d \geq 2$. Recall that P_{p^d} is generated by the elements g_1, \dots, g_d introduced in (3). Moreover, P_{p^d} acts imprimitively on the set $\{1, \dots, p^d\}$, a system of imprimitivity being given by $\Delta := \{\Delta_1, \dots, \Delta_p\}$, where $\Delta_s := \{(s-1)p^{d-1} + 1, \dots, sp^{d-1}\}$, for $s \in \{1, \dots, p\}$. Since E acts transitively on $\{1, \dots, p^d\}$, there is some $g \in E$ such that $g(1) = p^{d-1} + 1$; in particular, $g \cdot \Delta_1 = \Delta_2$. Since $p^{d-1} + 1 \neq 1$, we have $g \neq 1$, hence g is an element of order p . Moreover, the group $\langle g \rangle$ acts on Δ , so that we obtain a group homomorphism $\varphi : \langle g \rangle \rightarrow \mathfrak{S}(\Delta) \cong \mathfrak{S}_p$. Since $g \cdot \Delta_1 = \Delta_2 \neq \Delta_1$, φ must be injective. Thus $\varphi(g)$ has order p , implying $g \cdot \Delta_1 = \Delta_2$, $g \cdot \Delta_2 = \Delta_{i_3}, \dots, g \cdot \Delta_{i_p} = \Delta_1$, for $\{1, 2, i_3, \dots, i_p\} = \{1, \dots, p\}$.

Let $R := {}^\sigma P_{p^{d-1}} \leq P$, for some $\sigma \in \mathfrak{S}_n$. By Lemma 3.6, we know that $R = g_d^i P_{p^{d-1}} g_d^{-i}$, for some $i \in \{0, \dots, p-1\}$. Thus $\text{supp}(R) = \Delta_{i+1}$. So, for $s \in \{0, \dots, p-1\}$, the group $g^s R$ has support $g^s \cdot \Delta_{i+1}$. As we have just seen, the sets $\Delta_{i+1}, g \cdot \Delta_{i+1}, \dots, g^{p-1} \cdot \Delta_{i+1}$ are pairwise disjoint. Consequently, the groups $R, {}^g R, \dots, {}^{g^{p-1}} R$ are precisely the different subgroups of P_{p^d} that are P_{p^d} -conjugate to $P_{p^{d-1}}$. $B := \prod_{s=0}^{p-1} g^s R$ is the base group of P_{p^d} , and is contained in P . Clearly $g \notin B$, since $g(1) \notin \Delta_1$. Since $[P_{p^d} : B] = p$, this implies $P_{p^d} = \langle B, g \rangle \leq P \leq P_{p^d}$, and the proof is complete. \square

3.9 Lemma. *Let $n, t \in \mathbb{N}$ and let $m_1, \dots, m_t \in \mathbb{N}_0$ be such that $m_t \neq 0$ and $n = \sum_{i=1}^t m_i p^i$. Suppose that $m_1 = 1$ and $t \geq 2$. Let P be a maximal subgroup of $E(m_1, \dots, m_t)$ such that $E_{p,1} \not\leq P$. Then P contains a subgroup $Q \leq \prod_{i=2}^t \prod_{j=1}^{m_i} E_{p^i, j}$ that acts fixed point freely on $\{p+1, \dots, n\}$.*

Proof. For convenience, set $E' := \prod_{i=2}^t \prod_{j=1}^{m_i} E_{p^i, j}$, so that $E(m_1, \dots, m_t) = E_p \times E' \geq P$. By Goursat's Lemma, we may identify P with the quintuple $(P_1, K_1, \eta, P_2, K_2)$, where P_1 and P_2 are the projections of P onto E_p and onto E' , respectively, $K_1 := \{g \in E_p : (g, 1) \in P\} \trianglelefteq P_1$, $K_2 := \{h \in E' : (1, h) \in P\}$, and $\eta : P_2/K_2 \rightarrow P_1/K_1$ is a group isomorphism. Since $|E_p| = p$, there are precisely three possibilities for the section (P_1, K_1) of E_p :

- (i) $P_1 = K_1 = E_p$,
- (ii) $P_1 = K_1 = \{1\}$,
- (iii) $P_1 = E_p$ and $K_1 = \{1\}$.

Case (i) cannot occur, since we are assuming $E_p \not\leq P$. In case (ii) we get $P = E'$, so that the assertion then holds with $Q := P$. So suppose that $P_1 = E_p$ and $K_1 = \{1\}$, so that also $[P_2 : K_2] = p$. Next recall that $P/(K_1 \times K_2) \cong P_1/K_1 \cong P_2/K_2$; see, for instance, [2, 2.3.21]. This forces $|E'| = |P| = |K_2| \cdot |P_1| = |K_1| \cdot |P_2| = |P_2|$. Thus $P_2 = E'$, and K_2 is a maximal subgroup of E' . Assume that K_2 has a fixed point x on $\{p+1, \dots, n\}$. Then $x \in \text{supp}(E_{p^i,j})$, for some $i \geq 2$ with $m_i \neq 0$ and some $j \in \{1, \dots, m_i\}$. But then K_2 has to fix the entire support of $E_{p^i,j}$, since $E_{p^i,j}$ acts regularly on its support. This implies $[P_2 : K_2] \geq p^i \geq p^2$, a contradiction. Consequently, K_2 must act fixed point freely on $\{p+1, \dots, n\}$, and the assertion of the lemma follows with $Q := \{1\} \times K_2 \leq P$. \square

The next result will be one of the key ingredients of our proof of Theorem 1.1 in Section 4 below.

3.10 Proposition. *Let $n \in \mathbb{N}$ with p -adic expansion $n = p + \sum_{i=2}^r n_i p^i$, where $r \geq 2$ and $n_r \neq 0$. Let $Q \leq P_n$ be such that $P_{n-2p} \leq_{\mathfrak{S}_n} Q$ and $E(1, n_2, \dots, n_r) \leq_{\mathfrak{S}_n} Q$. Then $Q = P_n$.*

Proof. Let $2 \leq s \leq r$ be minimal such that $n_s \neq 0$. Then $n - 2p$ has p -adic expansion $n - 2p = \sum_{j=1}^{s-1} (p-1)p^j + (n_s - 1)p^s + \sum_{i=s+1}^r n_i p^i$. Moreover, we have

$$P_n = P_{p,1} \times \prod_{i=s}^r \prod_{j=1}^{n_i} P_{p^i,j} \quad \text{and} \quad E_n = E(1, n_2, \dots, n_r) = E_{p,1} \times \prod_{i=s}^r \prod_{j=1}^{n_i} E_{p^i,j}.$$

By our hypothesis, there is some $g \in \mathfrak{S}_n$ such that ${}^g E_{p,1} \times \prod_{i=s}^r \prod_{j=1}^{n_i} {}^g E_{p^i,j} \leq Q \leq P_n$. In consequence of Lemma 3.6 and Remark 3.7, we may suppose that ${}^g E_{p^i,j} \leq P_{p^i,j}$, for $i \geq 2$ and $1 \leq j \leq n_i$, as well as ${}^g E_{p,1} = E_{p,1} = P_{p,1}$. Since also $P_{n-2p} \leq_{\mathfrak{S}_n} Q$, there exists some $R \leq Q \leq P_n$ of the form

$$R = \prod_{i=1}^{s-1} \prod_{j=1}^{p-1} R_{p^i,j} \times \prod_{j=1}^{n_s-1} R_{p^s,j} \times \prod_{i=s+1}^r \prod_{j=1}^{n_i} R_{p^i,j},$$

where $R_{p^k,l} \leq_{\mathfrak{S}_n} P_{p^k,l}$, for all possible k and l . By Lemma 3.6 and Remark 3.7 again, we must have $\prod_{i=s+1}^r \prod_{j=1}^{n_i} R_{p^i,j} = \prod_{i=s+1}^r \prod_{j=1}^{n_i} P_{p^i,j} \leq P_n$. As well, there is some $k \in \{1, \dots, n_s\}$ and some $m \in \{1, \dots, p-1\}$ such that $\prod_{j=1}^{n_s-1} R_{p^s,j} = \prod_{j=1}^{k-1} P_{p^s,j} \times \prod_{l=k+1}^{n_s} P_{p^s,l} \leq P_n$ and $R_{p^{s-1},m} \leq P_{p^s,k}$. By Lemma 3.6, $R_{p^{s-1},m}$ is thus $P_{p^s,k}$ -conjugate to one of the p^{s-1} subgroups of $P_{p^s,k}$ that are \mathfrak{S}_n -conjugate to $P_{p^{s-1}}$. Since Q also contains the regular elementary abelian group ${}^g E_{p^s,k} \leq P_{p^s,k}$, Lemma 3.8 now implies that $P_{p^s,k} \leq Q$. Altogether this shows that indeed $P_n \leq Q$, and the assertion of the proposition follows. \square

4 The Proof of Theorem 1.1

The aim of this section is to establish a proof of Theorem 1.1. To this end, let F be a field of characteristic $p > 2$, and let $n \in \mathbb{N}$ be such that $n \equiv p \pmod{p^2}$. The simple $F\mathfrak{S}_n$ -module $D^{(n-p+1, 1^{p-1})}$ will henceforth be denoted by D . If $p = n$ then the Sylow p -subgroups of \mathfrak{S}_n

are abelian, and are thus the vertices of D , by Knörr's Theorem [8]. From now on we shall suppose that $n \geq p^2 + p$. Let P_n be the Sylow p -subgroup of \mathfrak{S}_n introduced in 3.2. In order to show that P_n is a vertex of D , we shall proceed as follows: suppose that $Q \leq P_n$ is a vertex of D . Then:

(i) Building on Wildon's result in [12, Theorem 2], it was shown in [4, Proposition 5.2] that $P_{n-2p} = P_{n-(p-1)-2} \times P_{p-1} <_{\mathfrak{S}_n} Q$.

(ii) Let $n = \sum_{i=2}^r n_i p^i + p$ be the p -adic expansion of n , where $r \geq 2$ and $n_r \neq 0$. We shall show in Proposition 4.7 below that $D(E(1, n_2, \dots, n_r)) \neq \{0\}$. Here $E(1, n_2, \dots, n_r)$ denotes the elementary abelian subgroup of P_n defined in 3.4, and $D(E(1, n_2, \dots, n_r))$ denotes the Brauer construction of D with respect to $E(1, n_2, \dots, n_r)$ as defined in 2.1. Thus, $E(1, n_2, \dots, n_r) \leq_{\mathfrak{S}_n} Q$, by [3, (1.3)].

(iii) Once we have verified (ii), we can apply Proposition 3.10, which will then show that $Q = P_n$.

4.1 Notation. (a) Let $\mathcal{B} := \mathcal{B}_{p-1}$ be the F -basis of D defined in (2), and let $u \in D$ be such that $u = \sum_{b \in \mathcal{B}} \lambda_b b$, for $\lambda_b \in F$. The basis element $e_2 \wedge e_3 \wedge \dots \wedge e_p \in \mathcal{B}$ will from now on be denoted by e . Moreover, suppose that $k, x \in \{2, \dots, n-1\}$ and that $k \leq p$. Then we denote the element $e_2 \wedge \dots \wedge e_{k-1} \wedge e_{k+1} \wedge \dots \wedge e_p \wedge e_x$ of D by $\hat{e}_k \wedge e_x$. In the case where $\hat{e}_k \wedge e_x \in \mathcal{B}$, the coefficient $\lambda_{e_2 \wedge \dots \wedge e_{k-1} \wedge e_{k+1} \wedge \dots \wedge e_p \wedge e_x}$ will be abbreviated by $\lambda_{\hat{k}, x}$.

Similarly, if $2 \leq k < l \leq p$ and if $x, y \in \{2, \dots, n-1\}$, then we set $\hat{e}_{k,l} \wedge e_x \wedge e_y := e_2 \wedge \dots \wedge e_{k-1} \wedge e_{k+1} \wedge \dots \wedge e_{l-1} \wedge e_{l+1} \wedge \dots \wedge e_p \wedge e_x \wedge e_y \in D$. In the case where $\hat{e}_{k,l} \wedge e_x \wedge e_y \in \mathcal{B}$, we denote by $\lambda_{\widehat{k,l,x,y}}$ the coefficient at $\hat{e}_{k,l} \wedge e_x \wedge e_y$ in u .

(b) Let $u \in D$ be such that $u = \sum_{b \in \mathcal{B}} \lambda_b b$, with $\lambda_b \in F$. We say that the basis element $b \in \mathcal{B}$ occurs in u with coefficient λ_b .

(c) For $k_1, k_2 \in \{2, \dots, n-1\}$, we set

$$s(k_1, k_2) := \begin{cases} k_2 - (k_1 - 1) & \text{if } k_1 \leq k_2, \\ 0 & \text{if } k_2 < k_1. \end{cases} \quad (4)$$

Thus, if $k_1 \leq k_2$ then

$$s(k_1, k_2) \equiv \begin{cases} 0 \pmod{2} & \text{if } k_1 \not\equiv k_2 \pmod{2}, \\ 1 \pmod{2} & \text{if } k_1 \equiv k_2 \pmod{2}. \end{cases}$$

(d) From now on, let $t, m_2, \dots, m_t \in \mathbb{N}$ be such that $t \geq 2$, $m_t \neq 0$, and $n = p + \sum_{i=2}^t m_i p^i$. The elementary abelian group $E(1, m_2, \dots, m_t) \leq \mathfrak{S}_n$ will be denoted by E . Note that, by our convention in 3.4, we have $(1, 2, \dots, p) \in E$. In the case where $t = r$ and $m_i = n_i$, for $i = 2, \dots, r$, we, in particular, get $E = E(1, n_2, \dots, n_r)$.

In the course of this section we shall have to compute explicitly the actions of elements in E on our chosen basis \mathcal{B} of D . The following lemmas will be used repeatedly in this section.

4.2 Lemma. Let $\alpha := (1, 2, \dots, p) \in \mathfrak{S}_n$. Let further $\beta := (x_1, \dots, x_p) \in \mathfrak{S}_n$ be such that $\{x_1, \dots, x_p\} \cap \{1, \dots, p\} = \emptyset$.

(a) For $i \in \{2, \dots, n-1\}$, one has

$$\alpha \cdot e_i = \begin{cases} e_{i+1} - e_2 & \text{if } 2 \leq i \leq p-1, \\ -e_2 & \text{if } i = p, \\ e_i - e_2 & \text{if } i \geq p+1. \end{cases}$$

(b) If $n \notin \text{supp}(\beta)$ then, for $i \in \{2, \dots, n-1\}$, one has

$$\beta \cdot e_i = \begin{cases} e_i & \text{if } i \notin \text{supp}(\beta), \\ e_{\beta(i)} & \text{if } i \in \text{supp}(\beta). \end{cases}$$

(c) If $x_p = n$ then, for $i \in \{2, \dots, n-1\}$, one has

$$\beta \cdot e_i = \begin{cases} e_i & \text{if } i \notin \text{supp}(\beta), \\ e_{\beta(i)} & \text{if } i \in \{x_1, \dots, x_{p-2}\}, \\ -\sum_{j=2}^{n-1} e_j & \text{if } i = x_{p-1}. \end{cases}$$

Proof. (a) If $2 \leq i \leq p-1$, then

$$\begin{aligned} \alpha \cdot e_i &= \alpha \cdot (\overline{\omega_i - \omega_1}) = \overline{\alpha \cdot (\omega_i - \omega_1)} = \overline{\omega_{\alpha(i)} - \omega_{\alpha(1)}} = \overline{\omega_{i+1} - \omega_2} = \overline{(\omega_{i+1} - \omega_1) - (\omega_2 - \omega_1)} \\ &= e_{i+1} - e_2. \end{aligned}$$

If $i = p$, then $\alpha \cdot e_i = \alpha \cdot (\overline{\omega_p - \omega_1}) = \overline{\omega_1 - \omega_2} = -e_2$. Finally, if $i \geq p+1$, then we have

$$\alpha \cdot e_i = \overline{\omega_i - \omega_2} = \overline{(\omega_i - \omega_1) - (\omega_2 - \omega_1)} = e_i - e_2.$$

The proofs of (b) and (c) are similar, and are left to the reader. \square

4.3 Lemma. Let $k, l \in \{2, \dots, p\}$, and let $x \in \{p+1, \dots, n-1\}$. Then one has

- (a) $e_{k+1} \wedge \dots \wedge e_p \wedge e_2 \wedge \dots \wedge e_{k-1} \wedge e_x = (-1)^{s(k+1,p)(k-2)} \hat{e}_k \wedge e_x$;
- (b) $\hat{e}_k \wedge e_k = (-1)^{s(k+1,p)} e$;
- (c) if $k < l$ then $\hat{e}_{k,l} \wedge e_x \wedge e_l = (-1)^{s(l+1,p)+1} \hat{e}_k \wedge e_x$;
- (d) if $k < l$ then $\hat{e}_{k,l} \wedge e_x \wedge e_k = (-1)^{s(k+1,p)} \hat{e}_l \wedge e_x$.

Proof. (a) For $k \in \{2, \dots, p\}$ and $x \in \{p+1, \dots, n-1\}$, we have

$$\begin{aligned} & \overbrace{e_{k+1} \wedge \dots \wedge e_p}^{s(k+1,p)} \wedge \underbrace{e_2 \wedge \dots \wedge e_{k-1}}_{k-2} \wedge e_x \\ &= (-1)^{s(k+1,p)} e_2 \wedge e_{k+1} \wedge \dots \wedge e_p \wedge e_3 \wedge \dots \wedge e_{k-1} \wedge e_x = (-1)^{s(k+1,p)(k-2)} \hat{e}_k \wedge e_x. \end{aligned}$$

The proofs of (b), (c) and (d) are similar, and are left to the reader. \square

4.4 Corollary. For $e := e_2 \wedge e_3 \wedge \dots \wedge e_p$, we have $e \in D^{P_n}$; in particular, $e \in D^P$, for every $P \leq P_n$.

Proof. With the notation in 3.2 we have $P_n = P_p \times \prod_{i=2}^r (P_{p^i})^{n_i}$, and $P_p = \langle \alpha \rangle$, where $\alpha := (1, 2, \dots, p)$. If $\beta \in \prod_{i=2}^r (P_{p^i})^{n_i}$ then we clearly have $\beta \cdot e = e$. By Lemma 4.2 and Lemma 4.3(b), we also have

$$\alpha \cdot e = (e_3 - e_2) \wedge (e_4 - e_2) \wedge \dots \wedge (e_p - e_2) \wedge (-e_2) = (-1)^{s(3,p)+1} e = (-1)^2 e = e.$$

□

4.5 Lemma. Let $1 \neq \sigma \in E$, and let $q \in \mathbb{N}$ be such that

$$\sigma = (x_1^1, \dots, x_p^1) \cdots (x_1^q, \dots, x_p^q),$$

where $\{x_i^s : 1 \leq i \leq p, 1 \leq s \leq q\} = \text{supp}(\sigma) \subseteq \{p+1, \dots, n\}$ and $x_p^q = n$. Let further $u \in D$ be such that $u = \sum_{b \in \mathcal{B}} \lambda_b b$, for $\lambda_b \in F$. Suppose that $\sigma \cdot u = u$. Then one has the following:

- (a) $\sum_{k=2}^p (-1)^k \lambda_{\hat{k}, x_i^q} = 0$, for every $i \in \{1, \dots, p-1\}$;
- (b) $\sum_{k=2}^p (-1)^{k+1} \lambda_{\hat{k}, x_i^s} = \sum_{k=2}^p (-1)^{k+1} \lambda_{\hat{k}, x_1^s}$, for $i \in \{1, \dots, p\}$ and $1 \leq s \leq q-1$.

Proof. Let $x \in \{x_i^s : 1 \leq i \leq p, 1 \leq s \leq q-1\}$, and let $k \in \{2, \dots, p\}$. Suppose that $b \in \mathcal{B}$ is such that $\hat{e}_k \wedge e_x$ occurs with non-zero coefficient in $\sigma \cdot b$. Then

- (i) $b = \hat{e}_k \wedge e_{\sigma^{-1}(x)}$, or
- (ii) $b = \hat{e}_k \wedge e_{x_{p-1}^q}$, or
- (iii) $b = \hat{e}_{k,k_2} \wedge e_{\sigma^{-1}(x)} \wedge e_{x_{p-1}^q}$ and $\sigma^{-1}(x) < x_{p-1}^q$, for some $k < k_2 \leq p$, or
- (iv) $b = \hat{e}_{k_1,k} \wedge e_{\sigma^{-1}(x)} \wedge e_{x_{p-1}^q}$ and $\sigma^{-1}(x) < x_{p-1}^q$, for some $2 \leq k_1 < k$, or
- (v) $b = \hat{e}_{k,k_2} \wedge e_{x_{p-1}^q} \wedge e_{\sigma^{-1}(x)}$ and $\sigma^{-1}(x) > x_{p-1}^q$, for some $k < k_2 \leq p$, or
- (vi) $b = \hat{e}_{k_1,k} \wedge e_{x_{p-1}^q} \wedge e_{\sigma^{-1}(x)}$ and $\sigma^{-1}(x) > x_{p-1}^q$, for some $2 \leq k_1 < k$.

If b is one of the basis elements in (i)–(vi) then the following table records $\sigma \cdot b$ as well as the coefficient at $\hat{e}_k \wedge e_x$ in $\sigma \cdot b$, which is obtained using Lemma 4.3.

b	$\sigma \cdot b$	coefficient
$\hat{e}_k \wedge e_{\sigma^{-1}(x)}$	$\hat{e}_k \wedge e_x$	1
$\hat{e}_k \wedge e_{x_{p-1}^q}$	$\hat{e}_k \wedge \sum_{y=2}^{n-1} (-e_y)$	-1
$\hat{e}_{k,k_2} \wedge e_{\sigma^{-1}(x)} \wedge e_{x_{p-1}^q}$	$\hat{e}_{k,k_2} \wedge e_x \wedge \sum_{y=2}^{n-1} (-e_y)$	$(-1)^{1+s(k_2+1,p)+1}$
$\hat{e}_{k_1,k} \wedge e_{\sigma^{-1}(x)} \wedge e_{x_{p-1}^q}$	$\hat{e}_{k_1,k} \wedge e_x \wedge \sum_{y=2}^{n-1} (-e_y)$	$(-1)^{1+s(k_1+1,p)}$
$\hat{e}_{k,k_2} \wedge e_{x_{p-1}^q} \wedge e_{\sigma^{-1}(x)}$	$\hat{e}_{k,k_2} \wedge \sum_{y=2}^{n-1} (-e_y) \wedge e_x$	$(-1)^{1+s(k_2+1,p)}$
$\hat{e}_{k_1,k} \wedge e_{x_{p-1}^q} \wedge e_{\sigma^{-1}(x)}$	$\hat{e}_{k_1,k} \wedge \sum_{y=2}^{n-1} (-e_y) \wedge e_x$	$(-1)^{s(k_1+1,p)}$

Now note that $(-1)^{1+s(k_2+1,p)+1} = (-1)^{1+p-k_2+1} = (-1)^{k_2+1}$ and $(-1)^{1+s(k_1+1,p)} = (-1)^{1+p-k_1} = (-1)^{k_1}$. Since $\sigma \cdot u = u$, this shows that

$$\lambda_{\hat{k},x} = \lambda_{\hat{k},\sigma^{-1}(x)} - \lambda_{\hat{k},x_{p-1}^q} + \sum_{k_2=k+1}^p (-1)^{k_2+1} \lambda_{\widehat{k,k_2,\sigma^{-1}(x),x_{p-1}^q}} + \sum_{k_1=2}^{k-1} (-1)^{k_1} \lambda_{\widehat{k_1,k,\sigma^{-1}(x),x_{p-1}^q}} \quad (5)$$

if $\sigma^{-1}(x) < x_{p-1}^q$ and

$$\lambda_{\hat{k},x} = \lambda_{\hat{k},\sigma^{-1}(x)} - \lambda_{\hat{k},x_{p-1}^q} - \sum_{k_2=k+1}^p (-1)^{k_2+1} \lambda_{\widehat{k,k_2,x_{p-1}^q,\sigma^{-1}(x)}} - \sum_{k_1=2}^{k-1} (-1)^{k_1} \lambda_{\widehat{k_1,k,x_{p-1}^q,\sigma^{-1}(x)}} \quad (6)$$

if $\sigma^{-1}(x) > x_{p-1}^q$. Moreover,

$$\begin{aligned} & \sum_{k=2}^p (-1)^{k+1} \left(\sum_{k_2=k+1}^p (-1)^{k_2+1} \lambda_{\widehat{k, k_2, \sigma^{-1}(x), x_{p-1}^q}} + \sum_{k_1=2}^{k-1} (-1)^{k_1} \lambda_{\widehat{k_1, k, \sigma^{-1}(x), x_{p-1}^q}} \right) \\ &= \sum_{k=2}^p \sum_{l=k+1}^p ((-1)^{k+1} (-1)^{l+1} + (-1)^k (-1)^{l+1}) \lambda_{\widehat{k, l, \sigma^{-1}(x), x_{p-1}^q}} = 0 \end{aligned}$$

if $\sigma^{-1}(x) < x_{p-1}^q$, and

$$\begin{aligned} & \sum_{k=2}^p (-1)^{k+1} \left(- \sum_{k_2=k+1}^p (-1)^{k_2+1} \lambda_{\widehat{k, k_2, x_{p-1}^q, \sigma^{-1}(x)}} - \sum_{k_1=2}^{k-1} (-1)^{k_1} \lambda_{\widehat{k_1, k, x_{p-1}^q, \sigma^{-1}(x)}} \right) \\ &= - \sum_{k=2}^p \sum_{l=k+1}^p ((-1)^{k+1} (-1)^{l+1} + (-1)^k (-1)^{l+1}) \lambda_{\widehat{k, l, x_{p-1}^q, \sigma^{-1}(x)}} = 0 \end{aligned}$$

if $\sigma^{-1}(x) > x_{p-1}^q$. Hence, from (5) and (6) we get

$$\sum_{k=2}^p (-1)^{k+1} \lambda_{\widehat{k, x_i^s}} = \sum_{k=2}^p (-1)^{k+1} \lambda_{\widehat{k, \sigma^{-1}(x_i^s)}} + \sum_{k=2}^p (-1)^k \lambda_{\widehat{k, x_{p-1}^q}}, \quad (7)$$

for every $i \in \{1, \dots, p\}$ and $1 \leq s \leq q-1$.

We also have $\sigma^i \cdot u = u$, for $i = 1, \dots, p-1$. To compare the coefficient at e in u and in $\sigma^i \cdot u$, let $i \in \{1, \dots, p-1\}$ and suppose that $b \in \mathcal{B}$ is such that e occurs in $\sigma^i \cdot b$ with non-zero coefficient. Then either $b = e$ and $e = \sigma^i \cdot e$, or $b = \hat{e}_k \wedge e_{\sigma^{-i}(x_p^q)}$, for some $k \in \{2, \dots, p\}$. Moreover, in the latter case we have $\sigma^i \cdot b = \hat{e}_k \wedge (-e_2 - e_3 - \dots - e_{n-1})$, where e occurs with coefficient

$$(-1)^{s(k+1, p)+1} = \begin{cases} 1 & \text{if } 2 \mid k, \\ -1 & \text{if } 2 \nmid k, \end{cases}$$

by Lemma 4.3. So we obtain $\lambda_e = \lambda_e + \sum_{k=2}^p (-1)^k \lambda_{\widehat{k, \sigma^{-i}(x_p^q)}}$, for $i \in \{1, \dots, p-1\}$, that is,

$$0 = \sum_{k=2}^p (-1)^k \lambda_{\widehat{k, x_j^q}}, \quad (8)$$

for $j \in \{1, \dots, p-1\}$, which proves assertion (a). Now assertion (b) follows from (7) and (8) with $j = p-1$. \square

Next we shall show that $D(E) \neq \{0\}$, where E is the elementary abelian group in 4.1. In order to do so, we want to apply Proposition 2.2 with $b_0 = e$.

4.6 Lemma. *Let P be a maximal subgroup of E . If $u \in D^P$ then e occurs in $\text{Tr}_P^E(u)$ with coefficient 0.*

Proof. Set $\alpha := (1, 2, \dots, p)$. Let $u \in D^P$, and write $u = \sum_{b \in \mathcal{B}} \lambda_b b$, where $\lambda_b \in F$. We shall treat the case where $\alpha \in P$ and the case where $\alpha \notin P$ separately.

Case 1: $\alpha \in P$. Then there is some $1 \neq g \in \prod_{i=2}^t \prod_{j=1}^{m_i} E_{p^i, j}$ with $g \notin P$. Thus $\{1, g, g^2, \dots, g^{p-1}\}$ is a set of representatives of the left cosets of P in E , so that we get $\text{Tr}_P^E(u) = u + g \cdot u + \dots + g^{p-1} \cdot u = \sum_{b \in \mathcal{B}} \sum_{i=0}^{p-1} \lambda_b(g^i \cdot b)$.

Since $g \neq 1$ and $t \geq 2$, we have

$$g = (x_1^1, \dots, x_p^1) \cdots (x_1^q, \dots, x_p^q),$$

for some $q \geq p$ and $\{x_i^s : 2 \leq i \leq p, 1 \leq s \leq q\} = \text{supp}(g)$.

Suppose first that $n \notin \text{supp}(g)$, and let $b \in \mathcal{B}$. Let further $i \in \{0, \dots, p-1\}$, and suppose that e occurs in $g^i \cdot b$ with non-zero coefficient. Then we must have $b = e$, in which case $\sum_{i=0}^{p-1} g^i \cdot b = pe = 0$, by Corollary 4.4; in particular, e occurs in $\text{Tr}_P^E(u)$ with coefficient 0.

So we may now suppose that $n \in \text{supp}(g)$. Moreover, we may suppose that $x_p^q = n$. Let $i \in \{0, \dots, p-1\}$, and let $b \in \mathcal{B}$ be such that e occurs in $g^i \cdot b$ with non-zero coefficient. If $i = 0$ then we must of course have $b = e = g^0 \cdot e$. If $i \geq 1$ then $b = e$, or $b = \hat{e}_k \wedge e_{g^{-i}(x_p^q)}$, for some $k \in \{2, \dots, p\}$. In the latter case, we have $g^i \cdot (\hat{e}_k \wedge e_{g^{-i}(x_p^q)}) = \hat{e}_k \wedge (-e_2 - e_3 - \dots - e_{n-1})$, in which e occurs with coefficient

$$(-1)^{s(k+1, p)+1} = \begin{cases} 1 & \text{if } 2 \mid k, \\ -1 & \text{if } 2 \nmid k, \end{cases}$$

by Lemma 4.3. Consequently, the coefficient at e in $\text{Tr}_P^E(u)$ equals

$$p\lambda_e + \sum_{i=1}^{p-1} \left(\sum_{\substack{k=2 \\ 2 \mid k}}^p \lambda_{\hat{k}, x_i^q} - \sum_{\substack{l=2 \\ 2 \nmid l}}^p \lambda_{\hat{l}, x_i^q} \right) = \sum_{i=1}^{p-1} \sum_{k=2}^p (-1)^k \lambda_{\hat{k}, x_i^q}. \quad (9)$$

Next we use the fact that $u \in D^P$ to show that this coefficient is indeed 0. Since $\alpha \in P$, we, in particular, have $u = \alpha^i \cdot u$, for every $i \in \{1, \dots, p-1\}$. So let $i \in \{1, \dots, p-1\}$, and let $x \in \{x_1^q, \dots, x_p^q\}$. Suppose that $b \in \mathcal{B}$ is such that $\hat{e}_{i+1} \wedge e_x$ occurs in $\alpha^i \cdot b$ with non-zero coefficient. Then from Lemma 4.2 we deduce that $b = \hat{e}_{\alpha^{-i}(1)} \wedge e_x$. Moreover, we have

$$\alpha^i \cdot (\hat{e}_{\alpha^{-i}(1)} \wedge e_x) = (e_{i+2} - e_{i+1}) \wedge \dots \wedge (e_p - e_{i+1}) \wedge (e_2 - e_{i+1}) \wedge \dots \wedge (e_i - e_{i+1}) \wedge (e_x - e_{i+1}).$$

Thus, by Lemma 4.3, the coefficient at $\hat{e}_{i+1} \wedge e_x$ in $\alpha^i \cdot (\hat{e}_{\alpha^{-i}(1)} \wedge e_x)$ equals $(-1)^{s(i+2, p)(i-1)} = 1$. Letting i vary over $\{1, \dots, p-1\}$ and comparing the coefficient at $\hat{e}_{i+1} \wedge e_x$ in u and in $\alpha^i \cdot u$, we deduce that $\lambda_{\hat{k}, x} = \lambda_{\widehat{p-k+2, x}}$, for $k \in \{2, \dots, (p+1)/2\}$ and every $x \in \{x_1^q, \dots, x_p^q\}$. Since k is even if and only if $p-k+2$ is odd, we conclude that the right-hand side of (9) is 0, as claimed. This completes the proof in case 1.

Case 2: $\alpha \notin P$, so that $\{1, \alpha, \alpha^2, \dots, \alpha^{p-1}\}$ is a set of representatives for the cosets of P in E , and we get $\text{Tr}_P^E(u) = u + \alpha \cdot u + \dots + \alpha^{p-1} \cdot u$. We determine the coefficient at e in $\text{Tr}_P^E(u) = u + \alpha \cdot u + \dots + \alpha^{p-1} \cdot u$. Let $i \in \{0, \dots, p-1\}$, and let $b \in \mathcal{B}$ be such that e occurs in $\alpha^i \cdot b$ with non-zero coefficient. If $i = 0$ then $b = e = \alpha^0 \cdot e$. So let $i \geq 1$. Then, by Lemma 4.2, we either have $b = e$, or $b = \hat{e}_{\alpha^{-i}(1)} \wedge e_x$, for some $x \in \{p+1, \dots, n-1\}$. Moreover, in the latter case,

$$\alpha^i \cdot b = (e_{i+2} - e_{i+1}) \wedge (e_{i+3} - e_{i+1}) \wedge \dots \wedge (e_p - e_{i+1}) \wedge (e_2 - e_{i+1}) \wedge \dots \wedge (e_i - e_{i+1}) \wedge (e_x - e_{i+1}).$$

So the coefficient at e in $\alpha^i \cdot (\hat{e}_{\alpha^{-i}(1)} \wedge e_x)$ equals

$$(-1)^{s(i+2,p)(i-1)+s(i+2,p)+1} = \begin{cases} 1 & \text{if } 2 \nmid i, \\ -1 & \text{if } 2 \mid i. \end{cases}$$

Since i is even if and only if $\alpha^{-i}(1)$ is even, we deduce from this that the coefficient at e in $u + \alpha \cdot u + \dots + \alpha^{p-1} \cdot u$ equals

$$p\lambda_e + \sum_{x=p+1}^{n-1} \sum_{k=2}^p (-1)^{k+1} \lambda_{\hat{k},x} = \sum_{x=p+1}^{n-1} \sum_{k=2}^p (-1)^{k+1} \lambda_{\hat{k},x}. \quad (10)$$

To show that this coefficient is 0, we again exploit the fact that $u \in D^P$. In fact, we shall show that

$$\sum_{\substack{x \in \text{supp}(E_{p^l, j_l}) \\ x < n}} \sum_{k=2}^p (-1)^{k+1} \lambda_{\hat{k},x} = 0, \quad (11)$$

for every $l \in \{2, \dots, t\}$ and $1 \leq j_l \leq m_l$. For each such l and j_l , there is, by Lemma 3.9, some element $\sigma(l, j_l) \in P$ such that $\text{supp}(E_{p^l, j_l}) \subseteq \text{supp}(\sigma(l, j_l)) \subseteq \{p+1, \dots, n\}$. Fixing l and j_l , we write

$$\sigma := \sigma(l, j_l) = (x_1^1, \dots, x_p^1) \cdots (x_1^q, \dots, x_p^q),$$

for some $q \geq |E_{p^l, j_l}|/p$ and $\text{supp}(\sigma) = \{x_i^j : 1 \leq i \leq p, 1 \leq j \leq q\}$.

Case 2.1: $n \notin \text{supp}(\sigma)$, or equivalently, $\text{supp}(\sigma) \cap \text{supp}(E_{p^t, m_t}) = \emptyset$. Let $x \in \text{supp}(\sigma)$, let $k \in \{2, \dots, p\}$, and let $b \in \mathcal{B}$ be such that $\hat{e}_k \wedge e_x$ occurs in $\sigma \cdot b$ with non-zero coefficient. This forces $b = \hat{e}_k \wedge e_{\sigma^{-1}(x)}$, and $\sigma \cdot (\hat{e}_k \wedge e_{\sigma^{-1}(x)}) = \hat{e}_k \wedge e_x$. Thus, $\lambda_{\hat{k},x} = \lambda_{\hat{k},\sigma^{-1}(x)}$. This shows that $\lambda_{\hat{k},x_1^s} = \lambda_{\hat{k},x_i^s}$, for all $i \in \{1, \dots, p\}$ and $s \in \{1, \dots, q\}$. By rearranging commuting p -cycles in σ , we may assume that there is some $1 \leq q_0 \leq q$ such that $\text{supp}(E_{p^l, j_l}) = \{x_i^s : 1 \leq i \leq p, 1 \leq s \leq q_0\}$. Then

$$\sum_{x \in \text{supp}(E_{p^l, j_l})} \sum_{k=2}^p (-1)^{k+1} \lambda_{\hat{k},x} = \sum_{i=1}^p \sum_{s=1}^{q_0} \sum_{k=2}^p (-1)^{k+1} \lambda_{\hat{k},x_i^s} = p \sum_{s=1}^{q_0} \sum_{k=2}^p (-1)^{k+1} \lambda_{\hat{k},x_1^s} = 0, \quad (12)$$

as desired.

Case 2.2: $n \in \text{supp}(\sigma)$. Then we may suppose that $x_p^q = n$. If $(l, j_l) \neq (t, m_t)$, then we may further suppose that there is some $1 \leq q_1 < q$ such that $\text{supp}(E_{p^l, j_l}) = \{x_i^s : 1 \leq i \leq p, 1 \leq s \leq q_1\}$. By Lemma 4.5(b), we then get

$$\sum_{x \in \text{supp}(E_{p^l, j_l})} \sum_{k=2}^p (-1)^{k+1} \lambda_{\hat{k},x} = \sum_{i=1}^p \sum_{s=1}^{q_1} \sum_{k=2}^p (-1)^{k+1} \lambda_{\hat{k},x_i^s} = p \sum_{s=1}^{q_1} \sum_{k=2}^p (-1)^{k+1} \lambda_{\hat{k},x_1^s} = 0. \quad (13)$$

If $(l, j_l) = (t, m_t)$ then we may suppose that there is $1 \leq q_2 \leq q$ such that $\text{supp}(E_{p^t, m_t}) =$

$\{x_i^s : 1 \leq i \leq p, q_2 \leq s \leq q\}$. In this case, Lemma 4.5 gives

$$\begin{aligned} \sum_{\substack{x \in \text{supp}(E_{p^t, m_t}) \\ x < n}} \sum_{k=2}^p (-1)^{k+1} \lambda_{\hat{k}, x} &= \sum_{i=1}^p \sum_{s=q_2}^{q-1} \sum_{k=2}^p (-1)^{k+1} \lambda_{\hat{k}, x_i^s} + \sum_{i=1}^{p-1} \sum_{k=2}^p (-1)^{k+1} \lambda_{\hat{k}, x_i^q} \\ &= p \sum_{s=q_2}^{q-1} \sum_{k=2}^p (-1)^{k+1} \lambda_{\hat{k}, x_1^s} - \sum_{i=1}^{p-1} \sum_{k=2}^p (-1)^k \lambda_{\hat{k}, x_i^q} = 0. \end{aligned}$$

To summarize, we have now verified equation (11), which together with (9) shows that the coefficient at e in $\text{Tr}_P^E(u)$ is 0. This now completes the proof in case 2 and, thus, of the lemma. \square

As a direct consequence of Lemma 4.6, Corollary 4.4, Proposition 2.2, and [3, (1.3)] we thus have proved the following

4.7 Proposition. *Let $n \in \mathbb{N}$ be such that $n = p + \sum_{i=2}^t m_i p^i$, for some $t \geq 2$, $m_2, \dots, m_t \in \mathbb{N}_0$ with $m_t \neq 0$. Let further $D := D^{(n-p+1, 1^{p-1})}$, and let $Q \leq \mathfrak{S}_n$ be a vertex of D . Then $D(E(1, m_2, \dots, m_t)) \neq \{0\}$; in particular, $E(1, m_2, \dots, m_t) \leq_{\mathfrak{S}_n} Q$.*

4.8 Remark. Again consider the p -adic expansion $n = p + \sum_{i=2}^r n_i p^i$, where $r \geq 2$ and $n_r \neq 0$. Note that Proposition 4.7, in particular, holds for $t = r$ and $m_1 = 1, \dots, m_r = n_r$. Thus the elementary abelian subgroup $E(1, n_2, \dots, n_r) \leq P_n$ is \mathfrak{S}_n -conjugate to a subgroup of every vertex of D . This settles item (ii) at the beginning of this section and completes the proof of Theorem 1.1.

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